

Strong Functional Representation Lemma and Applications to Coding Theorems

Cheuk Ting Li and Abbas El Gamal

Department of Electrical Engineering, Stanford University

Email: ctli@stanford.edu, abbas@ee.stanford.edu

Abstract

This paper shows that for any random variables X and Y , it is possible to represent Y as a function of (X, Z) such that Z is independent of X and $I(X; Z|Y) \leq \log(I(X; Y) + 1) + 4$. We use this strong functional representation lemma (SFRL) to establish a tighter bound on the rate needed for one-shot exact channel simulation than was previously established by Harsha et. al., and to establish achievability results for one-shot variable-length lossy source coding, multiple description coding and Gray-Wyner system. We also show that the SFRL can be used to reduce the channel with state noncausally known at the encoder to a point-to-point channel, which provides a simple achievability proof of the Gelfand-Pinsker theorem. Finally we present an example in which the SFRL inequality is tight to within 5 bits.

Index Terms

Functional representation lemma, channel simulation, one-shot achievability, lossy source coding, channel with state.

I. INTRODUCTION

The functional representation lemma [1, p. 626] states that for any random variables X and Y , there exists a random variable Z independent of X such that Y can be represented as a function of X and Z . This result has been used to establish several results in network information theory beginning with the early work of Hajek and Pursley on the broadcast channel [2] and Willems and van der Meulen on the multiple access channel with cribbing encoders [3]. In this paper, we strengthen this result by showing that for any X and Y , there exists a Z independent of X such that Y is a function of X and Z , and

$$I(X; Z|Y) \leq \log(I(X; Y) + 1) + 4.$$

We use this strong functional representation lemma (SFRL) together with an optimal prefix code such as a Huffman code to establish one-shot, *variable-length* coding results for channel simulation [4], Shannon's lossy source coding [5], multiple description coding [6], [7] and lossy Gray-Wyner system [8]. We then show how the SFRL can be used to reduce the channel with state known at the encoder to a point-to-point channel, providing a simple proof to the Gelfand-Pinsker theorem [9]. The asymptotic block coding counterparts of these one-shot results can be readily obtained by converting the variable-length code into a block code and incurring an error probability that vanishes as the block length approaches infinity.

A weaker form of the SFRL can be obtained using the result by Harsha et. al. [4] on the one-shot exact channel simulation with unlimited common randomness. Assuming the input X has a given pmf, then [4] implies that

$$I(X; Z|Y) \leq (1 + \epsilon) \log(I(X; Y) + 1) + c_\epsilon$$

is achievable, where $\epsilon > 0$ and c_ϵ is a function of ϵ . This result was established using a rejection sampling scheme and applies only to discrete random variables. In comparison SFRL uses a construction that we refer to as Poisson functional representation which provides a tighter bound and applies to arbitrary random variables.

One-shot achievability results using fixed length (random) coding have been recently established for lossy source coding and several setting in network information theory. In [10], Liu, Cuff and Verdú established a one-shot achievability result for lossy source coding using channel resolvability. One-shot quantum lossy source coding settings were investigated by Datta et. al. [11]. In [12], Verdú introduced non-asymptotic packing and covering lemmas and used them to establish one-shot achievability results for several settings including Gelfand-Pinsker. In [13], Liu, Cuff and Verdú proved a one-shot mutual covering lemma and used it to establish a one-shot achievability result for the broadcast channel. In [14], Watanabe, Kuzuoka and Tan established several one-shot achievability results for coding with side-information (including Gelfand-Pinsker). In [15], Yassaee, Aref and Gohari established several one-shot achievability results, including Gelfand-Pinsker and multiple description coding. Most of these results are stated in terms of information density and various other quantities. In contrast, our one-shot achievability results using variable-length codes are all stated in terms only of mutual information. Moreover, given the SFRL, our proofs are generally simpler.

Variable-length (asymptotic or finite blocklength) lossy source coding settings have been studied, e.g., see [16], [17], [18], [19]. Most of these works concern the universal setting in which the distribution of the source is unknown, hence the use of variable-length codes is justified. In contrast, the reason we consider variable-length code in this paper is that it allows us to give one-shot results that subsume their asymptotic fixed-length counterparts.

In the following section, we state the SFRL and prove it for discrete random variable Y . In Sections III and IV we use SFRL to establish one-shot achievability results for channel simulation and three source coding settings, respectively. In Section V, we use SFRL together with Shannon's channel coding theorem to provide a simple achievability proof of the Gelfand–Pinsker theorem. In Section VI we extend the proof of the SFRL to general random variables. Finally in Section VII we demonstrate the tightness of the SFRL inequality and discuss several other properties of this inequality.

Notation

Throughout this paper, we assume that \log is base 2 and the entropy H is in bits. The binary entropy function is $H(x) = -x \log x - (1-x) \log(1-x)$. We use the notation: $X_a^b = (X_a, \dots, X_b)$, $X^n = X_1^n$, $[a : b] = [a, b] \cap \mathbb{Z}$ and $[a] = [1 : a]$.

For discrete X , we write the probability mass function as p_X . For continuous X , we write the probability density function as f_X . For general random variable X , we write the probability measure (push-forward measure by X) as \mathbf{P}_X .

II. STRONG FUNCTIONAL REPRESENTATION LEMMA

The main result in this paper is given in the following.

Theorem 1 (Strong functional representation lemma). *For any pair of random variables $(X, Y) \sim \mathbf{P}_{XY}$ with $I(X; Y) < \infty$, there exists a random variable Z independent of X such that Y can be expressed as a function $g(X, Z)$ of X and Z , and*

$$I(X; Z|Y) \leq \log(I(X; Y) + 1) + 4.$$

Moreover, if X and Y are discrete with cardinalities $|\mathcal{X}|$ and $|\mathcal{Y}|$, respectively, then $|\mathcal{Z}| \leq |\mathcal{X}|(|\mathcal{Y}| - 1) + 2$.

Note that SFRL can be applied conditionally; given $\mathbf{P}_{XY|U}$, we can represent Y as a function $g(X, Z, U)$ such that Z is independent of (X, U) and

$$I(X; Z|Y, U) \leq \log(I(X; Y|U) + 1) + 4.$$

The reason we can have a Z independent of U is that by the functional representation lemma, we can represent Z as a function of U and \tilde{Z} such that \tilde{Z} is independent of U and use \tilde{Z} instead of Z .

Note that SFRL applies to general distributions \mathbf{P}_{XY} . Although $H(Y)$ may be infinite, the cardinality of Y conditioned on Z can still be countable and $H(Y|Z)$ can be finite. Note that $Z \perp\!\!\!\perp X$ and $H(Y|X, Z) = 0$ imply that $I(X; Z|Y) = H(Y|Z) - I(X; Y)$. Hence the SFRL implies the existence of a $Z \perp\!\!\!\perp X$ such that $H(Y|Z)$ is close to $I(X; Y)$.

For simplicity of presentation, we first prove the SFRL for discrete Y . The proof is extended to general Y in Section VI. To prove the SFRL, we use the following random variable Z and function g construction.

Definition 1 (Exponential functional representation). Let X and Y be random variables, where $Y \in \{1, \dots, |\mathcal{Y}|\}$ and $|\mathcal{Y}|$ is finite or countably infinite. The exponential functional representation of Y given X is defined as $Y = g_{X \rightarrow Y}(X, Z^{|\mathcal{Y}|})$, where $Z^{|\mathcal{Y}|}$ is a sequence of i.i.d. $\text{Exp}(1)$ random variables independent of X , and

$$g_{X \rightarrow Y}(x, z^{|\mathcal{Y}|}) = \arg \min_{y \in \mathcal{Y}} \frac{z_y}{p_{Y|X}(y|x)}.$$

Note that if $|\mathcal{Y}|$ is finite, we can generate $Z^{|\mathcal{Y}|}$ uniformly over the probability simplex on \mathcal{Y} . This is equivalent to the original scheme after normalization such that $\sum_y Z_y = 1$. We now proceed to prove Theorem 1 for discrete Y by showing that the exponential functional representation satisfies the constraints.

Proof: Let $\Phi_y = Z_y/p_Y(y)$,

$$\Theta = \inf_y \frac{Z_y}{p_{Y|X}(y|x)},$$

and K be the index of Φ_Y in $\{\Phi_y\}_{y \in \mathcal{Y}}$ sorted in ascending order (hence $|\{y : \Phi_y < \Phi_Y\}| = K - 1$ with probability 1). Since Y is a function of $Z^{|\mathcal{Y}|}$ and K , we have $H(Y|Z^{|\mathcal{Y}|}) \leq H(K)$. We now proceed to bound $H(K)$. Let $r(x, y) = \frac{p_{Y|X}(y|x)}{p_Y(y)}$. Since $\Theta|\{X = x, Y = y\} \sim \text{Exp}(1)$,

$$\begin{aligned} \mathbf{E}[\log K | X = x] &= \sum_y p_{Y|X}(y|x) \mathbf{E}[\log K | X = x, Y = y] \\ &= \sum_y p(y|x) \int_0^\infty e^{-\theta} \mathbf{E}[\log K | X = x, Y = y, \Theta = \theta] d\theta. \end{aligned}$$

Consider

$$\begin{aligned} &\mathbf{E}[\log K | X = x, Y = y, \Theta = \theta] \\ &= \mathbf{E} \left[\log \left(\left| \left\{ y' \neq y : \Phi_{y'} < \frac{\theta p(y|x)}{p(y)} \right\} \right| + 1 \right) \mid \frac{Z_{y'}}{p(y'|x)} \geq \theta \text{ for all } y' \right] \end{aligned}$$

$$\begin{aligned}
&\leq \log \left(\mathbb{E} \left[\left| \left\{ y' \neq y : \Phi_{y'} < \frac{\theta p(y|x)}{p(y)} \right\} \right| \middle| \frac{Z_{y'}}{p(y'|x)} \geq \theta \text{ for all } y' \right] + 1 \right) \\
&= \log \left(\sum_{y' \neq y} \mathbb{P} \left\{ \Phi_{y'} < \frac{\theta p(y|x)}{p(y)} \middle| \frac{Z_{y'}}{p(y'|x)} \geq \theta \right\} + 1 \right) \\
&= \log \left(\sum_{y' \neq y} \mathbb{P} \left\{ Z_{y'} < \frac{\theta p(y|x)p(y')}{p(y)} \middle| Z_{y'} \geq \theta p(y'|x) \right\} + 1 \right) \\
&= \log \left(\sum_{y': r(x, y') < r(x, y)} (1 - \exp(-\theta p(y') (r(x, y) - r(x, y')))) + 1 \right) \\
&\leq \log \left(\sum_{y': r(x, y') < r(x, y)} \theta p(y') (r(x, y) - r(x, y')) + 1 \right) \\
&\leq \log (\theta r(x, y) + 1).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E} [\log K | X = x] &\leq \sum_y p(y|x) \int_0^\infty e^{-\theta} \log (\theta r(x, y) + 1) d\theta \\
&\leq \sum_y p(y|x) \log (r(x, y) + 1) \\
&= \sum_{y: r(x, y) \geq 1} p(y|x) \log (r(x, y) + 1) + \sum_{y: r(x, y) < 1} p(y|x) \log (r(x, y) + 1) \\
&\leq \sum_{y: r(x, y) \geq 1} p(y|x) (\log r(x, y) + 1) + \sum_{y: r(x, y) < 1} p(y|x) \\
&= \sum_{y: r(x, y) \geq 1} p(y|x) \log r(x, y) + 1 \\
&= D(p_{Y|X}(\cdot|x) \| p_Y) - \sum_{y: r(x, y) < 1} p(y|x) \log r(x, y) + 1 \\
&\leq D(p_{Y|X}(\cdot|x) \| p_Y) + e^{-1} \log e + 1,
\end{aligned}$$

where the last inequality follows from Appendix A in [4]. Therefore $\mathbb{E} [\log K] \leq I(X; Y) + e^{-1} \log e + 1$. Note that by the maximum entropy distribution subject to a given $\mathbb{E} [\log K]$, we have

$$H(K) \leq \mathbb{E} [\log K] + \log (\mathbb{E} [\log K] + 1) + 1.$$

The proof of this bound is given in the Appendix for the sake of completeness. Hence

$$\begin{aligned}
H(K) &\leq I(X; Y) + e^{-1} \log e + 2 + \log (I(X; Y) + e^{-1} \log e + 2) \\
&\leq I(X; Y) + \log (I(X; Y) + 1) + e^{-1} \log e + 2 + \log (e^{-1} \log e + 2) \\
&< I(X; Y) + \log (I(X; Y) + 1) + 4.
\end{aligned}$$

To prove the cardinality bound, first note that if $|\mathcal{X}|, |\mathcal{Y}|$ are finite, then $|\mathcal{Z}| \leq |\mathcal{Y}|^{|\mathcal{X}|}$ can be assumed to be finite since it is the number of different functions $x \mapsto g_{X \rightarrow Y}(x, z)$ for different z . To further reduce the cardinality, we apply Carathéodory's theorem on the $(|\mathcal{X}|(|\mathcal{Y}| - 1) + 1)$ -dimensional vectors with entries $H(Y|Z = z)$ and $p(x, y|z)$ for $x \in \{1, \dots, |\mathcal{X}|\}$, $y \in \{1, \dots, |\mathcal{Y}| - 1\}$; see [20], [21]. ■

III. ONE-SHOT CHANNEL SIMULATION

Channel simulation aims to find the minimum amount of communication over a noiseless channel needed to simulate a memoryless channel $\mathbf{P}_{Y|X}$. Several settings of this problem have been studied, e.g., see [22], [23], [24]. Consider the one-shot channel simulation with unlimited common randomness setup [4] in which Alice and Bob share unlimited common randomness W . Alice observes $X \sim \mathbf{P}_X$ and sends a prefix-free description M to Bob via a noiseless channel such that Bob can generate Y (from M and W) according to a prescribed conditional distribution $\mathbf{P}_{Y|X}$. The problem is to find the minimum expected description length of M , $\mathbb{E} [L(M)]$, needed. It is straightforward to show that $\mathbb{E} [L(M)] \geq I(X; Y)$. In [4], it is shown that for X and Y discrete,

$$\mathbb{E} [L(M)] \leq I(X; Y) + (1 + \epsilon) \log(I(X; Y) + 1) + c_\epsilon$$

is achievable for $\epsilon > 0$, where c_ϵ is a function of ϵ .

We now show that the SFRL provides a tighter upper bound on $\mathbb{E}[L(M)]$ that applies to arbitrary (not only discrete) memoryless channels. By the SFRL, there exists a Z independent of X such that $Y = g_{X \rightarrow Y}(X, Z)$ and

$$H(Y|Z) \leq I(X; Y) + \log(I(X; Y) + 1) + 4.$$

We use $W = Z$ as the common randomness. Upon observing $X = x$, Alice generates $Y \sim \mathbf{P}_{Y|X}(\cdot|x)$ and encodes Y using a Huffman code for the pmf $p_{Y|Z}(\cdot|Z)$ into the description M (note that Y can be arbitrary but by the SFRL $Y|Z = z$ is discrete). Bob then recovers Y from M and Z . The expected length is

$$\mathbb{E}[L(M)] \leq I(X; Y) + \log(I(X; Y) + 1) + 5.$$

In Section VII, we give an example which shows that the $\log(I(X; Y) + 1)$ term is in general necessary.

Moreover, for discrete X, Y , the amount of the common randomness can be bounded by $\log|\mathcal{W}| \leq \log(|\mathcal{X}|(|\mathcal{Y}| - 1) + 2)$. In comparison, the amount of the common randomness in [4] can be bounded by $O(\log(|\mathcal{X}||\mathcal{Y}|))$ only if the expected description length is increased by $O(\log \log(|\mathcal{X}| + |\mathcal{Y}|))$.

We can use the exponential functional representation to construct $W = Z^{|\mathcal{Y}|}$ (if $|\mathcal{W}|$ is unlimited). Upon observing W and X , Alice generates $Y \sim \mathbf{P}_{Y|X}$ and finds K , the index of $Z_Y/p_Y(Y)$ in the set $\{Z_y/p_Y(y)\}_{y \in \mathcal{Y}}$ sorted in ascending order as in the proof of the SFRL, and encodes K into M using the optimal prefix-free code for the power-law distribution $q(k) \propto k^{-\lambda}$, $k = 1, 2, \dots$, where $\lambda = 1 + 1/(I(X; Y) + e^{-1} \log e + 1)$. Bob recovers K from M and obtains Y using K and $Z^{|\mathcal{Y}|}$.

Remark 1. In [4], the setting in which $X = x$ is an arbitrary input (instead of $X \sim p_X$) is studied. It is shown that

$$\mathbb{E}[L(M)] \leq C + (1 + \epsilon) \log(C + 1) + c_\epsilon$$

for all $x \in \mathcal{X}$ is achievable, where C is the capacity of the channel $p_{Y|X}$ and c_ϵ is a function of ϵ .

The exponential functional representation can still be applied to this setting. If we encode K (defined in the proof of the SFRL) into M using the optimal prefix-free code for the power-law distribution $q(k) \propto k^{-\lambda}$, where $\lambda = 1 + 1/(C + e^{-1} \log e + 1)$, then by the same argument in the proof of the SFRL, and Claim 3.1 in [4],

$$\mathbb{E}[L(M)] \leq C + \log(C + 1) + 5$$

is achievable.

IV. LOSSY SOURCE CODING

We use the SFRL to establish one-shot achievability results for three lossy source coding settings.

A. Lossy source coding

Consider the following one-shot variable-length lossy source coding problem. We are given a random variable (source) $X \in \mathcal{X}$ with $X \sim \mathbf{P}_X$, a reproduction alphabet \mathcal{Y} , and a distortion function $d : \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$ (note that X, Y can be arbitrary, and $d(x, y)$ can be infinite). Given X , the encoder selects $\tilde{Y} \in \mathcal{Y}$ and encodes it using a prefix-free code into $M \in \{0, 1\}^*$. The decoder recovers \tilde{Y} from M . Let $\bar{R} = \mathbb{E}[L(M)]$ be the expected value of the length of the description M and $\mathbb{E}[d(X, \tilde{Y})]$ be the average distortion of representing X by \tilde{Y} . An expected length-distortion pair (\bar{R}, D) is said to be achievable if there exists a variable-length code with expected description length \bar{R} such that $\mathbb{E}[d(X, \tilde{Y})] \leq D$.

In the following we use the SFRL to establish a set of achievable (\bar{R}, D) pairs.

Theorem 2. A pair (\bar{R}, D) is achievable for the one-shot variable-length lossy source coding problem with source $X \sim \mathbf{P}_X$, reproduction alphabet \mathcal{Y} , and distortion measure $d(x, y)$ if

$$\bar{R} > R(D) + \log(R(D) + 1) + 6,$$

where

$$R(D) = \inf_{\mathbf{P}_{Y|X} : \mathbb{E}[d(X, Y)] \leq D} I(X; Y)$$

is the (asymptotic) rate-distortion function [5].

Proof: Let Y be the random variable that attains $\mathbb{E}[d(X, Y)] \leq D$ and $I(X; Y) \leq R(D) + \epsilon$. By the SFRL, there exists Z independent of X such that $Y = g_{X \rightarrow Y}(X, Z)$ and

$$H(g_{X \rightarrow Y}(X, Z)|Z) \leq I(X; Y) + \eta,$$

where $\eta = \log(I(X; Y) + 1) + 4$. Consider the set

$$A = \{(H(g_{X \rightarrow Y}(X, z)), \mathbb{E}_X[d(X, g_{X \rightarrow Y}(X, z))]) : z \in \mathcal{Z}\}.$$

Since $(H(g_{X \rightarrow Y}(X, Z)|Z), \mathbf{E}[d(X, Y)])$ is a weighted average of the points in A , it is in the convex hull of A . By Carathéodory's theorem, there exists z_0, z_1 and $\lambda \in [0, 1]$ such that

$$(1 - \lambda)H(g_{X \rightarrow Y}(X, z_0)) + \lambda H(g_{X \rightarrow Y}(X, z_1)) \leq H(g_{X \rightarrow Y}(X, Z)|Z) \leq I(X; Y) + \eta,$$

$$(1 - \lambda)\mathbf{E}_X[d(X, g_{X \rightarrow Y}(X, z_0))] + \lambda \mathbf{E}_X[d(X, g_{X \rightarrow Y}(X, z_1))] \leq \mathbf{E}[d(X, Y)].$$

Note that to satisfy the above inequalities, we need one point less than stated in Carathéodory's theorem. Take $Q \sim \text{Bern}(\lambda)$, $\tilde{Y} = g_{X \rightarrow Y}(X, z_Q)$. Then

$$H(\tilde{Y}) \leq H(\tilde{Y}|Q) + 1 \leq I(X; Y) + \eta + 1.$$

We use a Huffman code to encode \tilde{Y} and obtain an expected length $\bar{R} \leq H(\tilde{Y}) + 1$. The result follows by letting $\epsilon \rightarrow 0$. ■

Although the above achievability proof does not use random coding, it can be interpreted as using the following *soft random coding* scheme (for discrete Y).

Soft codebook generation. The random variable Z produced by the SFRL in the proof of Theorem 2 represents the choice of the codebook. We select a “soft codebook” by fixing $Z^{|\mathcal{Y}|} = z^{|\mathcal{Y}|}$. Unlike conventional codebook $\mathcal{C} \subseteq Y$ in which each y can either be in \mathcal{C} or not, a soft codebook $z^{|\mathcal{Y}|}$ assigns to each y a weight $w_y = p_Y(y)/z_y$, which indicates the likelihood that y is used.

Encoding. The encoder observes x and finds

$$\tilde{y} = \arg \max_y w_y \cdot \frac{p_{Y|X}(y|x)}{p_Y(y)}.$$

It then finds k , the index of $w_{\tilde{y}}$ in $\{w_y\}_{y \in \mathcal{Y}}$ sorted in descending order, and encodes k using an optimal prefix-free code for the power-law distribution $q(k) \propto k^{-\lambda}$, where $\lambda = 1 + 1/(I(X; Y) + e^{-1} \log e + 1)$. This is analogous to the case in conventional codebook generation in which we find the closest $\tilde{y} \in \mathcal{C}$ to x and encodes it into its index in \mathcal{C} . Here we use a prefix-free code over the positive integers to encode the index into the description m because the number of possible codewords \tilde{y} (which is typically the entire \mathcal{Y}) is large, but those with large w_y are more likely to be used so they are assigned shorter descriptions.

Decoding. The decoder receives m , recovers k , then finds \tilde{y} at the index k in $\{w_y\}_{y \in \mathcal{Y}}$ sorted in descending order.

The finite blocklength variable-length lossy source coding problem [16] concerns the case in which the source is memoryless and average per symbol distortion $d(x^n, y^n) = (1/n) \sum_i d(x_i, y_i)$. In [25] it is shown that the expected per symbol description length $\bar{R}/n = R(D) + (1 + o(1))(1/n) \log n$ is achievable via d -semifaithful codes [26] with $d(X^n, \tilde{Y}^n) \leq D$ surely. Applying Theorem 2 to X^n , we have

$$\bar{R}/n = R(D) + (1/n)(\log(nR(D) + 1) + 6) = R(D) + (1 + o(1))(1/n) \log n.$$

Hence we achieve the same redundancy as [25] albeit under the expected distortion constraint instead of the stronger sure distortion constraint using the d -semifaithful codes.

We can use Theorem 2 to establish the achievability of Shannon's (asymptotic) lossy source coding theorem [5], assuming there exists a symbol $y_0 \in \mathcal{Y}$ with finite $d(x, y_0)$ for all x . First note that the redundancy $(1 + o(1))(1/n) \log n$ in the finite block length extension can be made arbitrarily small, hence \bar{R}/n can be made arbitrarily close to $R(D)$. Now we use the finite block length scheme over l blocks of n source symbols each of length n (for a total block length of nl). By the law of large numbers, the probability that the total description length is greater than $nl(R(D) + \epsilon)$ tends to 0 as the block length approaches infinity. Hence, we can construct a fixed length code out of the variable-length code by simply discarding descriptions longer than $nl(R(D) + \epsilon)$ and assigning the reconstruction sequence (y_0, \dots, y_0) to the discarded descriptions.

B. Multiple Description Coding

In this section, we use the SFRL to establish a one-shot inner bound for the variable-length multiple description coding problem, which yields an alternative proof of the El Gamal-Cover inner bound [6] and the Zhang-Berger inner bound [7], [27], [28] in the asymptotic regime. The encoder observes $X \sim \mathbf{P}_X$ and produces two prefix-free descriptions $M_1, M_2 \in \{0, 1\}^*$. Decoder 1 observes M_1 and generates \tilde{Y}_1 with distortion $d_1(X, \tilde{Y}_1)$. Similarly, Decoder 2 observes M_2 and produces \tilde{Y}_2 with distortion $d_2(X, \tilde{Y}_2)$. Decoder 0 observes M_1 and M_2 and produces \tilde{Y}_0 with distortion $d_0(X, \tilde{Y}_0)$. An expected description length-distortion tuple $(\bar{R}_1, \bar{R}_2, D_0, D_1, D_2)$ is said to be achievable if there exists a scheme with expected description length $\mathbf{E}[L(M_i)] \leq \bar{R}_i$ and expected distortion $\mathbf{E}[d_i(X, \tilde{Y}_i)] \leq D_i$.

Theorem 3. *The tuple $(\bar{R}_1, \bar{R}_2, D_0, D_1, D_2)$ is achievable if*

$$\begin{aligned} \bar{R}_1 &\geq I(X; Y_1, U) + 2\eta, \\ \bar{R}_2 &\geq I(X; Y_2, U) + 2\eta, \\ \bar{R}_1 + \bar{R}_2 &\geq I(X; Y_0, Y_1, Y_2|U) + 2I(X; U) + I(Y_1; Y_2|U) + 5\eta, \\ D_i &\geq \mathbf{E}[d_i(X, Y_i)] \text{ for } i = 0, 1, 2 \end{aligned}$$

for some $P_{U,Y_0,Y_1,Y_2|X}$, where

$$\eta = \log(I(X; Y_0, Y_1, Y_2, U) + I(Y_1; Y_2|U) + 1) + 7.$$

Note that the only difference between the above region and Zhang-Berger inner bound is the addition of η , which grows like $\log n$ if we consider X^n and does not affect the asymptotic rate.

Proof: It suffices to prove the achievability of the corner point:

$$\begin{aligned}\bar{R}_1 &= I(X; Y_1|U) + I(X; U) + 2\eta - 1, \\ \bar{R}_2 &= I(X, Y_1; Y_2|U) + I(X; Y_0|Y_1, Y_2, U) + I(X; U) + 3\eta - 1, \\ D_i &= \mathbf{E}[d_i(X, Y_i)] \text{ for } i = 0, 1, 2.\end{aligned}$$

The desired rate region can be achieved by time sharing between this corner point and the other corner point where Y_1, Y_2 are flipped, resulting in a penalty of at most 1 bit (we can use the first bits of M_1 and M_2 to represent which corner point it is).

Applying the SFRL to X, U , we have $U = g_{X \rightarrow U}(X, Z_3)$, where $Z_3 \perp\!\!\!\perp X$ such that

$$\begin{aligned}H(U|Z_3) &\leq I(X; U) + \log(I(X; U) + 1) + 4 \\ &\leq I(X; U) + \eta - 3.\end{aligned}$$

Applying the SFRL to X, Y_1 conditioned on U , we have $Y_1 = g_{X \rightarrow Y_1|U}(X, Z_1, U)$, where $Z_1 \perp\!\!\!\perp (X, U)$ such that

$$\begin{aligned}H(Y_1|U, Z_1) &\leq I(X; Y_1|U) + \log(I(X; Y_1|U) + 1) + 4 \\ &\leq I(X; Y_1|U) + \eta - 3.\end{aligned}$$

Applying the SFRL to $(X, Y_1), Y_2$ conditioned on U , we have $Y_2 = g_{X Y_1 \rightarrow Y_2|U}(X, Y_1, Z_2, U)$, $Z_2 \perp\!\!\!\perp (X, Y_1, U)$ such that

$$\begin{aligned}H(Y_2|U, Z_2) &\leq I(X, Y_1; Y_2|U) + \log(I(X, Y_1; Y_2|U) + 1) + 4 \\ &\leq I(X, Y_1; Y_2|U) + \eta - 3.\end{aligned}$$

Applying the SFRL to X, Y_0 conditioned on (Y_1, Y_2, U) , we have $Y_0 = g_{X \rightarrow Y_0|Y_1 Y_2 U}(X, Z_0, Y_1, Y_2, U)$, $Z_0 \perp\!\!\!\perp (X, Y_1, Y_2, U)$ such that

$$\begin{aligned}H(Y_0|Y_1, Y_2, U, Z_0) &\leq I(X; Y_0|Y_1, Y_2, U) + \log(I(X; Y_0|Y_1, Y_2, U) + 1) + 4 \\ &\leq I(X; Y_0|Y_1, Y_2, U) + \eta - 3.\end{aligned}$$

Note that $Z_0^3 \perp\!\!\!\perp X$. Consider the convex hull of the 7-dimensional vectors

$$\begin{bmatrix} H(U|Z_0^3 = z_0^3) \\ H(Y_1|U, Z_0^3 = z_0^3) \\ H(Y_2|U, Z_0^3 = z_0^3) \\ H(Y_0|Y_1, Y_2, U, Z_0^3 = z_0^3) \\ \mathbf{E}[d_0(X, Y_0) | Z_0^3 = z_0^3] \\ \mathbf{E}[d_1(X, Y_1) | Z_0^3 = z_0^3] \\ \mathbf{E}[d_2(X, Y_2) | Z_0^3 = z_0^3] \end{bmatrix}$$

for different $z_0^3 \in \mathcal{Z}_0 \times \mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3$. By Carathéodory's theorem, there exists a pmf p_Q with cardinality $|\mathcal{Q}| \leq 7$ and $\tilde{z}_0^3(q)$ such that

$$H(U|Q, Z_0^3 = \tilde{z}_0^3(Q)) \leq I(X; U) + \eta - 3,$$

and similarly for the other 6 inequalities. Take $\tilde{U} = g_{X \rightarrow U}(X, \tilde{z}_3(Q))$, $\tilde{Y}_1 = g_{X \rightarrow Y_1|U}(X, \tilde{z}_1(Q), \tilde{U})$, $\tilde{Y}_2 = g_{X Y_1 \rightarrow Y_2|U}(X, \tilde{Y}_1, \tilde{z}_2(Q), \tilde{U})$ and $\tilde{Y}_0 = g_{X \rightarrow Y_0|Y_1 Y_2 U}(X, \tilde{z}_0(Q), \tilde{Y}_1, \tilde{Y}_2, \tilde{U})$. Write $C_{p_Y}(y) \in \{0, 1\}^*$ for the Huffman codeword of y for the distribution p_Y . We set M_1 to be the concatenation of Q (3 bits), $C_{p_{\tilde{U}|Q}}(\cdot|Q)(\tilde{U})$ and $C_{p_{\tilde{Y}_1|\tilde{U}Q}}(\cdot|\tilde{U}, Q)(\tilde{Y}_1)$, and M_2 to be the concatenation of Q , $C_{p_{\tilde{U}|Q}}(\cdot|Q)(\tilde{U})$, $C_{p_{\tilde{Y}_2|\tilde{U}Q}}(\cdot|\tilde{U}, Q)(\tilde{Y}_2)$ and $C_{p_{\tilde{Y}_0|\tilde{Y}_1 \tilde{Y}_2 \tilde{U}Q}}(\cdot|\tilde{Y}_1, \tilde{Y}_2, \tilde{U}, Q)(\tilde{Y}_0)$. The expected length of M_1 is upper bounded by

$$\begin{aligned}3 + (I(X; U) + \eta - 3 + 1) + (I(X; Y_1|U) + \eta - 3 + 1) \\ = I(X; Y_1|U) + I(X; U) + 2\eta - 1.\end{aligned}$$

The bound on the expected length of M_2 can be obtained similarly.

Decoder 1 receives M_1 and recovers Q , and then recovers \tilde{U} by decoding the Huffman code for the distribution $p_{\tilde{U}|Q}(\cdot|Q)$, and then recovers \tilde{Y}_1 similarly. Decoder 2 receives M_2 and recovers Q, \tilde{U} and \tilde{Y}_2 . Decoder 0 receives M_1, M_2 and recovers $Q, \tilde{U}, \tilde{Y}_1, \tilde{Y}_2$ and \tilde{Y}_0 . ■

C. Lossy Gray–Wyner System

In this section, we use the SFRL to establish a one-shot inner bound for the lossy Gray–Wyner system [8], which yields an alternative proof of the achievability of the rate region in the asymptotic regime. The encoder observes $(X_1, X_2) \sim \mathbf{P}_{X_1, X_2}$ and produces three prefix-free descriptions $M_0, M_1, M_2 \in \{0, 1\}^*$. Decoder 1 observes M_0, M_1 and generates \tilde{Y}_1 with distortion $d_1(X_1, \tilde{Y}_1)$. Similarly, Decoder 2 observes M_0, M_2 and produces \tilde{Y}_2 with distortion $d_2(X_2, \tilde{Y}_2)$. An expected description length-distortion tuple $(\bar{R}_0, \bar{R}_1, \bar{R}_2, D_1, D_2)$ is said to be achievable if there exists a scheme with expected description length $\mathbb{E}[L(M_i)] \leq \bar{R}_i$ and expected distortion $\mathbb{E}[d_i(X_i, \tilde{Y}_i)] \leq D_i$.

Theorem 4. *The tuple $(\bar{R}_0, \bar{R}_1, \bar{R}_2, D_1, D_2)$ is achievable if*

$$\begin{aligned}\bar{R}_0 &\geq I(X_1, X_2; U) + \log(I(X_1, X_2; U) + 1) + 8, \\ \bar{R}_1 &\geq I(X_1; Y_1|U) + \log(I(X_1; Y_1|U) + 1) + 5, \\ \bar{R}_2 &\geq I(X_2; Y_2|U) + \log(I(X_2; Y_2|U) + 1) + 5, \\ D_i &\geq \mathbb{E}[d_i(X_i, Y_i)] \text{ for } i = 1, 2\end{aligned}$$

for some $\mathbf{P}_{U|X_1, X_2}, \mathbf{P}_{Y_1|X_1, U}, \mathbf{P}_{Y_2|X_2, U}$.

Note that the only difference between the above region and the lossy Gray–Wyner rate region [1, p. 357] is the addition of the logarithm terms, which grows like $\log n$ if we consider X_1^n, X_2^n and does not affect the asymptotic rate.

Proof: Applying the SFRL to $(X_1, X_2), U$, we have $U = g_{X_1 X_2 \rightarrow U}(X_1, X_2, Z_0)$, where $Z_0 \perp\!\!\!\perp (X_1, X_2)$ such that

$$H(U|Z_0) \leq I(X_1, X_2; U) + \log(I(X_1, X_2; U) + 1) + 4.$$

Applying the SFRL to X_1, Y_1 conditioned on U , we have $Y_1 = g_{X_1 \rightarrow Y_1|U}(X_1, Z_1, U)$, where $Z_1 \perp\!\!\!\perp (X_1, U)$ such that

$$H(Y_1|U, Z_1) \leq I(X_1; Y_1|U) + \log(I(X_1; Y_1|U) + 1) + 4.$$

Applying the SFRL to X_2, Y_2 conditioned on U , we have $Y_2 = g_{X_2 \rightarrow Y_2|U}(X_2, Z_2, U)$, where $Z_2 \perp\!\!\!\perp (X_2, U)$ such that

$$H(Y_2|U, Z_2) \leq I(X_2; Y_2|U) + \log(I(X_2; Y_2|U) + 1) + 4.$$

Note that $Z_0^2 \perp\!\!\!\perp (X_1, X_2)$. Consider the convex hull of the 5-dimensional vectors

$$\begin{bmatrix} H(U|Z_0^2 = z_0^2) \\ H(Y_1|U, Z_0^2 = z_0^2) \\ H(Y_2|U, Z_0^2 = z_0^2) \\ \mathbb{E}[d_1(X_1, Y_1) | Z_0^2 = z_0^2] \\ \mathbb{E}[d_2(X_2, Y_2) | Z_0^2 = z_0^2] \end{bmatrix}$$

for different $z_0^2 \in \mathcal{Z}_0 \times \mathcal{Z}_1 \times \mathcal{Z}_2$. By Carathéodory's theorem, there exists a pmf p_Q with cardinality $|\mathcal{Q}| \leq 5$ and $\tilde{z}_0^2(q)$ such that

$$H(U|Q, Z_0^2 = \tilde{z}_0^2(Q)) \leq I(X_1, X_2; U) + \log(I(X_1, X_2; U) + 1) + 4,$$

and similarly for the other 4 inequalities. Take $\tilde{U} = g_{X_1 X_2 \rightarrow U}(X_1, X_2, \tilde{z}_0(Q))$, $\tilde{Y}_1 = g_{X_1 \rightarrow Y_1|U}(X_1, \tilde{z}_1(Q), \tilde{U})$ and $\tilde{Y}_2 = g_{X_2 \rightarrow Y_2|U}(X_2, \tilde{z}_2(Q), \tilde{U})$. Write $C_{p_Y}(y) \in \{0, 1\}^*$ for the Huffman codeword of y for the distribution p_Y . We set M_0 to be the concatenation of Q (3 bits) and $C_{p_{\tilde{U}|Q}}(\cdot|Q)(\tilde{U})$, $M_1 = C_{p_{\tilde{Y}_1|\tilde{U}Q}}(\cdot|\tilde{U}, Q)(\tilde{Y}_1)$ and $M_2 = C_{p_{\tilde{Y}_2|\tilde{U}Q}}(\cdot|\tilde{U}, Q)(\tilde{Y}_2)$. The expected length of M_0 is upper bounded by

$$\begin{aligned} &3 + (H(U|Z_0) + 1) \\ &\leq 3 + (I(X_1, X_2; U) + \log(I(X_1, X_2; U) + 1) + 4 + 1) \\ &= I(X_1, X_2; U) + \log(I(X_1, X_2; U) + 1) + 8. \end{aligned}$$

The bound on the expected length of M_1, M_2 can be obtained similarly.

Decoder 1 receives M_0, M_1 and recovers Q , and then recovers \tilde{U} by decoding the Huffman code for the distribution $p_{\tilde{U}|Q}(\cdot|Q)$, and then recovers \tilde{Y}_1 by decoding the Huffman code for the distribution $p_{\tilde{Y}_1|\tilde{U}Q}(\cdot|\tilde{U}, Q)$. Similar for Decoder 2. ■

V. ACHIEVABILITY OF GELFAND–PINSKER

In this section, we use the SFRL to prove the achievability part of the Gelfand-Pinsker theorem [9] for discrete memoryless channels with discrete memoryless state $p_S p_{Y|X,S}$, where the state is noncausally available at the encoder. The asymptotic capacity of this setting is

$$C_{\text{GP}} = \max_{p_{U|S}, x(u,s)} (I(U; Y) - I(U; S)).$$

We show the achievability of any rate below C_{GP} directly by using the SFRL to reduce the channel to a point-to-point memoryless channel. Fix $p_{U|S}$ and $x(u, s)$ that attain the capacity. Applying the SFRL to S, U , there exists a random variable $V \perp\!\!\!\perp S$ such that

$$H(U|V) \leq I(U; S) + \log(I(U; S) + 1) + 4.$$

Note that

$$\begin{aligned} I(V; Y) &= I(U; Y) - I(U; Y|V) \\ &\geq I(U; Y) - H(U|V) \\ &\geq I(U; Y) - I(U; S) - \log(I(U; S) + 1) - 4. \end{aligned}$$

Hence we have constructed a memoryless point-to-point channel $p_{Y|V}$ with achievable rate close to $I(U; Y) - I(U; S)$.

For n channel uses, let $U^n | \{S^n = s^n\} \sim \prod_i p_{U|S}(u_i | s_i)$. The SFRL applied to S^n, U^n gives

$$I(V; Y^n) \geq nI(U; Y) - nI(U; S) - \log(nI(U; S) + 1) - 4.$$

Now we use the channel $p_{Y^n|V}$ l times (for a total block length of nl). By the channel coding theorem, we can communicate $l(nI(U; Y) - nI(U; S) - \log(nI(U; S) + 1) - 4) - o(l)$ bits with error probability that tends to 0 as $l \rightarrow \infty$. Letting $n \rightarrow \infty$ completes the proof.

In the above proof, we see that the SFRL can be used to convert a channel with state into a point-to-point channel by “orthogonalizing” the auxiliary input U and the state S . The point-to-point channel can be constructed explicitly via exponential functional representation. This construction can be useful for designing codes for channels with state based on codes for point-to-point channels. It is interesting to note that this reduction makes the achievability proof for the Gelfand–Pinsker quite similar to that for the causal case in which the channel is reduced to a point-to-point channel using the “Shannon strategy” (see [1, p. 176]).

Note that Marton’s inner bound for the broadcast channels with private messages [29] can also be proved using the SFRL in a similar manner. The idea is to “orthogonalize” the dependent auxiliary random variables U_1, U_2 by applying the SFRL on U_1, U_2 to produce two independent input random variables, and treat them with Y_1, Y_2 as an interference channel, and finally to treat interference as noise.

VI. GENERAL DISTRIBUTIONS AND POISSON FUNCTIONAL REPRESENTATION

In this section, we prove Theorem 1 for general X and Y (over a Polish space with Borel probability measure). We first extend the exponential functional representation to general distributions.

Definition 2 (Poisson functional representation). Fix any joint distribution \mathbf{P}_{XY} . Let $0 \leq T_1 \leq T_2 \leq \dots$ be a Poisson point process with rate 1 (i.e., the increments $T_i - T_{i-1}$ are i.i.d. $\text{Exp}(1)$ for $i = 1, 2, \dots$ with $T_0 = 0$), and $\tilde{Y}_1, \tilde{Y}_2, \dots$ be i.i.d. with $\tilde{Y}_1 \sim \mathbf{P}_Y$. Take $Z = \{(T_i, \tilde{Y}_i)\}_{i=1,2,\dots}$, i.e., a marked Poisson point process. Define

$$g_{X \rightarrow Y}(x, \{(t_i, \tilde{y}_i)\}) = \tilde{y} \left(\arg \min_i t_i \cdot \frac{d\mathbf{P}_Y}{d\mathbf{P}_{Y|X}(\cdot|x)}(\tilde{y}_i) \right),$$

where we write $\tilde{y}(k) = \tilde{y}_k$ for readability.

Note that if Y is discrete, the Poisson functional representation reduces to the exponential functional representation by letting

$$Z_y = p_Y(y) \cdot \min_{i: \tilde{Y}_i=y} T_i.$$

Now we proceed to prove Theorem 1 by showing that the Poisson functional representation satisfies the constraints.

Proof: Condition on the event $\{X = x\}$. First we show that $g_{X \rightarrow Y}(x, \{(T_i, \tilde{Y}_i)\})$ follows the distribution $\mathbf{P}_{Y|X}(\cdot|x)$. By the marking theorem of the Poisson point process [30], $\{(T_i, \tilde{Y}_i)\}$ is a Poisson point process over the product measure $\mu \times \mathbf{P}_Y$ (where μ is the Lebesgue measure on $[0, \infty)$). By the displacement theorem [30],

$$\left\{ \left(T_i \cdot \frac{d\mathbf{P}_Y}{d\mathbf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i), \tilde{Y}_i \right) \right\}$$

is a Poisson point process over $\mu \times \mathbf{P}_{Y|X}(\cdot|x)$. Hence

$$\min_i T_i \cdot \frac{d\mathbf{P}_Y}{d\mathbf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i) \sim \text{Exp}(1),$$

and

$$\tilde{Y} \left(\arg \min_i T_i \cdot \frac{d\mathbf{P}_Y}{d\mathbf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i) \right) \sim \mathbf{P}_{Y|X}(\cdot|x),$$

where we write $\tilde{Y}(k) = \tilde{Y}_k$. Now we bound $H(Y | \{(T_i, \tilde{Y}_i)\})$. Let

$$\begin{aligned} \Theta &= \min_i T_i \cdot \frac{d\mathbf{P}_Y}{d\mathbf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i), \\ K &= \arg \min_i T_i \cdot \frac{d\mathbf{P}_Y}{d\mathbf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i), \end{aligned}$$

then $H(Y | \{(T_i, \tilde{Y}_i)\}) \leq H(K)$. Conditioned on $\Theta = \theta$, $\tilde{Y}_K \sim \mathbf{P}_{Y|X}(\cdot|x)$ and $\{(T_i, \tilde{Y}_i)\}_{i \neq K}$ is a Poisson point process over the semidirect product measure

$$\nu(A \times B) = \int_B \mu \left(A \cap \left[\theta \cdot \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(y), \infty \right) \right) d\mathbf{P}_Y(y).$$

Note that $K - 1 = |\{i : T_i < T_K\}|$. Hence $K - 1$ conditioned on $\Theta = \theta$ and $\tilde{Y}_K = \tilde{y}$ follows the Poisson distribution with rate

$$\begin{aligned} \nu([0, T_K] \times \mathcal{Y}) &= \nu \left(\left[0, \theta \cdot \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(\tilde{y}) \right) \times \mathcal{Y} \right) \\ &= \int_{\mathcal{Y}} \mu \left(\left[0, \theta \cdot \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(\tilde{y}) \right) \cap \left[\theta \cdot \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(y), \infty \right) \right) d\mathbf{P}_Y(y) \\ &= \theta \int_{\mathcal{Y}} \max \left\{ 0, \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(\tilde{y}) - \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(y) \right\} d\mathbf{P}_Y(y) \\ &\leq \theta \int_{\mathcal{Y}} \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(\tilde{y}) \cdot d\mathbf{P}_Y(y) \\ &= \theta \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(\tilde{y}). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}[\log K] &= \mathbb{E}_{Y \sim \mathbf{P}_{Y|X}(\cdot|x)} \left[\int_0^\infty e^{-\theta} \mathbb{E}[\log K | \Theta = \theta, \tilde{Y}_K = Y] d\theta \right] \\ &\leq \mathbb{E}_{Y \sim \mathbf{P}_{Y|X}(\cdot|x)} \left[\int_0^\infty e^{-\theta} \log \left(\theta \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(Y) + 1 \right) d\theta \right] \\ &\leq \mathbb{E}_{Y \sim \mathbf{P}_{Y|X}(\cdot|x)} \left[\log \left(\int_0^\infty e^{-\theta} \theta \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(Y) d\theta + 1 \right) \right] \\ &= \mathbb{E}_{Y \sim \mathbf{P}_{Y|X}(\cdot|x)} \left[\log \left(\frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(Y) + 1 \right) \right] \\ &\leq \mathbb{E}_{Y \sim \mathbf{P}_{Y|X}(\cdot|x)} \left[\max \left\{ \log \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(Y), 0 \right\} + 1 \right] \\ &= D(\mathbf{P}_{Y|X}(\cdot|x) \| \mathbf{P}_Y) - \mathbb{E}_{Y \sim \mathbf{P}_{Y|X}(\cdot|x)} \left[\min \left\{ \log \frac{d\mathbf{P}_{Y|X}(\cdot|x)}{d\mathbf{P}_Y}(Y), 0 \right\} \right] + 1 \\ &\leq D(\mathbf{P}_{Y|X}(\cdot|x) \| \mathbf{P}_Y) + e^{-1} \log e + 1, \end{aligned}$$

where the last line follows by the same arguments as in Appendix A in [4]. For $X \sim \mathbf{P}_X$,

$$\mathbb{E}[\log K] \leq I(X; Y) + e^{-1} \log e + 1.$$

By the maximum entropy distribution subject to a given $\mathbb{E}[\log K]$ (see the Appendix), we have

$$\begin{aligned} H(K) &\leq I(X; Y) + e^{-1} \log e + 2 + \log(I(X; Y) + e^{-1} \log e + 2) \\ &\leq I(X; Y) + \log(I(X; Y) + 1) + e^{-1} \log e + 2 + \log(e^{-1} \log e + 2) \\ &< I(X; Y) + \log(I(X; Y) + 1) + 4. \end{aligned}$$

■

VII. TIGHTNESS OF THE SFRL

The SFRL states that $I(X; Z|Y) \leq \log(I(X; Y) + 1) + 4$ is achievable. A natural question to ask is whether this inequality is tight for some X, Y . In this section, we show that the log term is necessary and that the SFRL is in general tight to within 5 bits. Define the *excess functional information* as

$$\Psi(X \rightarrow Y) = \inf_{Z: Z \perp\!\!\!\perp X, H(Y|X, Z)=0} I(X; Z|Y).$$

Then by the SFRL, $\Psi(X \rightarrow Y) \leq \log(I(X; Y) + 1) + 4$. We now establish the following lower bound on $\Psi(X \rightarrow Y)$.

Proposition 1. *For discrete Y ,*

$$\Psi(X \rightarrow Y) \geq - \sum_{y \in \mathcal{Y}} \int_0^1 \mathbf{P}_X \{p_{Y|X}(y|X) \geq t\} \log(\mathbf{P}_X \{p_{Y|X}(y|X) \geq t\}) dt - I(X; Y).$$

Moreover for $|\mathcal{Y}| = 2$, equality holds in the above inequality, and the infimum in $\Psi(X \rightarrow Y)$ is attained via the exponential functional representation.

Proof: Fix $Z \perp\!\!\!\perp X$ such that $Y = g(X, Z)$. For any y , let $V_y = p_{Y|Z}(y|Z)$, $U \sim \text{Unif}[0, 1]$, $\tilde{X}_y = p_{Y|X}(y|X)$, $\tilde{V}_y = \mathbf{P} \{ \tilde{X}_y \geq U \mid U \}$, then $\mathbf{E}[V_y] = \mathbf{E}[\tilde{V}_y] = p_Y(y)$. We have

$$\begin{aligned} \int_v^1 \mathbf{P}\{V_y \geq t\} dt &= \mathbf{E}[\max\{V_y - v, 0\}] \\ &= \mathbf{E}_Z[\max\{p_{Y|Z}(y|Z) - v, 0\}] \\ &= \mathbf{E}_Z[\max\{\mathbf{P}_X\{g(X, Z) = y \mid Z\} - v, 0\}] \\ &= \mathbf{E}_Z[\max\{\mathbf{E}_{\tilde{X}_y}[\mathbf{P}_X\{g(X, Z) = y \mid Z, \tilde{X}_y\} \mid Z] - v, 0\}] \\ &= \mathbf{E}_Z[\max\{\mathbf{E}_{\tilde{X}_y}[\mathbf{P}_X\{g(X, Z) = y \mid Z, \tilde{X}_y\} \mid Z] - \mathbf{E}_{\tilde{X}_y}[\mathbf{1}\{\tilde{X}_y > F_{\tilde{X}_y}^{-1}(1 - v)\}], 0\}] \\ &\leq \mathbf{E}_Z[\mathbf{E}_{\tilde{X}_y}[\max\{\mathbf{P}_X\{g(X, Z) = y \mid Z, \tilde{X}_y\} - \mathbf{1}\{\tilde{X}_y > F_{\tilde{X}_y}^{-1}(1 - v)\}, 0\} \mid Z]] \\ &= \mathbf{E}_Z[\mathbf{E}_{\tilde{X}_y}[\mathbf{P}_X\{g(X, Z) = y \mid Z, \tilde{X}_y\} \cdot \mathbf{1}\{\tilde{X}_y \leq F_{\tilde{X}_y}^{-1}(1 - v)\} \mid Z]] \\ &= \mathbf{E}_{\tilde{X}_y}[\mathbf{E}_Z[\mathbf{P}_X\{g(X, Z) = y \mid Z, \tilde{X}_y\} \mid \tilde{X}_y] \cdot \mathbf{1}\{\tilde{X}_y \leq F_{\tilde{X}_y}^{-1}(1 - v)\}] \\ &= \mathbf{E}_{\tilde{X}_y}[\mathbf{E}_X[\mathbf{P}_Z\{g(X, Z) = y \mid X\} \mid \tilde{X}_y] \cdot \mathbf{1}\{\tilde{X}_y \leq F_{\tilde{X}_y}^{-1}(1 - v)\}] \\ &= \mathbf{E}_{\tilde{X}_y}[\mathbf{E}_X[p_{Y|X}(y|X) \mid \tilde{X}_y] \cdot \mathbf{1}\{\tilde{X}_y \leq F_{\tilde{X}_y}^{-1}(1 - v)\}] \\ &= \mathbf{E}_{\tilde{X}_y}[\tilde{X}_y \cdot \mathbf{1}\{\tilde{X}_y \leq F_{\tilde{X}_y}^{-1}(1 - v)\}] \\ &= \mathbf{E}_U[\max\{\mathbf{P}\{\tilde{X}_y \geq U \mid U\} - v, 0\}] \\ &= \mathbf{E}[\max\{\tilde{V}_y - v, 0\}] \\ &= \int_v^1 \mathbf{P}\{\tilde{V}_y \geq t\} dt. \end{aligned}$$

Hence V_y dominates \tilde{V}_y stochastically in the second order. By the concavity of $-t \log t$, we have

$$\begin{aligned} H(Y|Z) &= - \sum_y \mathbf{E}_Z[p_{Y|Z}(y|Z) \log p_{Y|Z}(y|Z)] \\ &= - \sum_y \mathbf{E}[V_y \log V_y] \\ &\geq - \sum_y \mathbf{E}[\tilde{V}_y \log \tilde{V}_y] \\ &= - \sum_y \int_0^1 \mathbf{P}_X\{p_{Y|X}(y|X) \geq u\} \log(\mathbf{P}_X\{p_{Y|X}(y|X) \geq u\}) du. \end{aligned} \tag{1}$$

Therefore,

$$I(X; Z|Y) \geq - \sum_y \int_0^1 \mathbf{P}_X\{p_{Y|X}(y|X) \geq t\} \log(\mathbf{P}_X\{p_{Y|X}(y|X) \geq t\}) dt - I(X; Y).$$

One can verify that for $|\mathcal{Y}| = 2$, equality in (1) holds by the definition of exponential functional representation. \blacksquare

The following proposition shows that there exists a sequence of (X, Y) for which the bound $\Psi(X, Y) \leq \log(I(X; Y) + 1) + 4$ given in the SFRL is tight within 5 bits.

Proposition 2. *For every $\alpha \geq 0$, there exists discrete X, Y such that $I(X; Y) \geq \alpha$ and*

$$\Psi(X \rightarrow Y) \geq \log(I(X; Y) + 1) - 1.$$

Proof: Let $k \in \{0, 1, \dots\}$, $V \in [0 : 2^k - 1]$,

$$p_V(v) = \gamma^{-1} 2^{k - \lceil \log(v+1) \rceil},$$

where $\gamma = 2^{k-1}(k+2)$, and let $X \sim \text{Unif}[0 : 2^k - 1]$, $Y = (X + V) \bmod 2^k$. Note that $|\{v : \gamma p_V(v) > t\}| = \gamma p_V(\lfloor t \rfloor)$ for $t \geq 0$. We have

$$\begin{aligned} & - \sum_{y \in \mathcal{Y}} \int_0^1 \mathbf{P}_X \{p_{Y|X}(y|X) \geq t\} \log(\mathbf{P}_X \{p_{Y|X}(y|X) \geq t\}) dt \\ &= - \sum_{y \in \mathcal{Y}} \int_0^1 (2^{-k} |\{v : p_V(v) \geq t\}|) \log(2^{-k} |\{v : p_V(v) \geq t\}|) dt \\ &= k - \int_0^1 |\{v : p_V(v) \geq t\}| \log |\{v : p_V(v) \geq t\}| dt \\ &= k - \int_0^1 \gamma p_V(\lfloor \gamma t \rfloor) \log(\gamma p_V(\lfloor \gamma t \rfloor)) dt \\ &= k - \sum_{v=0}^{2^k-1} p_V(v) \log(\gamma p_V(v)) dt \\ &= k - \log \gamma + H(V). \end{aligned}$$

And

$$I(X; Y) = H(Y) - H(Y|X) = k - H(V).$$

By Proposition 1,

$$\Psi(X \rightarrow Y) \geq k - \log \gamma + H(V) - (k - H(V)) = 2H(V) - \log \gamma.$$

One can check that

$$H(V) = \frac{1}{2}k + \log(k+2) - \frac{3}{2} + \frac{1}{k+2}.$$

Hence

$$I(X; Y) = \frac{1}{2}k - \log(k+2) + \frac{3}{2} - \frac{1}{k+2} \leq \frac{1}{2}k,$$

and

$$\begin{aligned} \Psi(X \rightarrow Y) &\geq k + 2 \log(k+2) - 3 + \frac{2}{k+2} - \log(2^{k-1}(k+2)) \\ &= \log(k+2) - 2 + \frac{2}{k+2} \\ &\geq \log(I(X; Y) + 1) - 1. \end{aligned}$$

Besides the upper bound given by the SFRL and its tightness, in the following we establish other properties of $\Psi(X \rightarrow Y)$. We write the conditional excess functional information as

$$\Psi(X \rightarrow Y | Q) = \mathbf{E}_Q[\Psi(X \rightarrow Y | Q = q)].$$

Proposition 3. *The excess functional information $\Psi(X \rightarrow Y)$ satisfies the following properties.*

1) *Alternative characterization.*

$$\Psi(X \rightarrow Y) = \inf_{Z: Z \perp\!\!\!\perp X} H(Y|Z) - I(X; Y).$$

2) *Monotonicity. If $X_1 \perp\!\!\!\perp X_2$ and $X_1 \perp\!\!\!\perp (X_2, Y_2) | Y_1$, then*

$$\Psi((X_1, X_2) \rightarrow (Y_1, Y_2)) \geq \Psi(X_1 \rightarrow Y_1).$$

3) *Subadditivity.* If $(X_1, Y_1) \perp\!\!\!\perp (X_2, Y_2)$, then

$$\Psi((X_1, X_2) \rightarrow (Y_1, Y_2)) \leq \Psi(X_1 \rightarrow Y_1) + \Psi(X_2 \rightarrow Y_2).$$

As a result, if we further have $X_2 \perp\!\!\!\perp Y_2$, then $\Psi((X_1, X_2) \rightarrow (Y_1, Y_2)) = \Psi(X_1 \rightarrow Y_1)$ by monotonicity.

4) *Data processing of $\Psi + I$.* If $X_2 - X_1 - Y_1 - Y_2$ forms a Markov chain,

$$\Psi(X_1 \rightarrow Y_1) + I(X_1; Y_1) \geq \Psi(X_2 \rightarrow Y_2) + I(X_2; Y_2).$$

5) *Upper bound by common entropy.*

$$\Psi(X \rightarrow Y) \leq G(X; Y) - I(X; Y) \leq \min \{H(X|Y), H(Y|X)\},$$

where $G(X; Y) = \min_{X \perp\!\!\!\perp Y|W} H(W)$ is the common entropy [31], [32].

6) *Conditioning.* If Q satisfies $H(Q|X) = 0$, then

$$\Psi(X \rightarrow Y) \geq \Psi(X \rightarrow Y | Q).$$

If we further have $H(Q|Y) = 0$, then equality holds in the above inequality.

7) *Successive minimization.*

$$\Psi(X \rightarrow Y) = \inf_{V: V \perp\!\!\!\perp X} \{I(X; V|Y) + \Psi(X \rightarrow Y | V)\}.$$

Proof:

1) *Alternative characterization.* Note that if $Z \perp\!\!\!\perp X$ and $H(Y|X, Z) = 0$, then $H(Y|Z) - I(X; Y) = I(X; Z|Y)$, hence

$$\inf_{Z: Z \perp\!\!\!\perp X, H(Y|X, Z)=0} I(X; Z|Y) \geq \inf_{Z: Z \perp\!\!\!\perp X} H(Y|Z) - I(X; Y).$$

For the other direction, assume $Z \perp\!\!\!\perp X$. By the functional representation lemma, let $Y = g(X, Z, \tilde{Z})$, $\tilde{Z} \perp\!\!\!\perp (X, Z)$. We have

$$\begin{aligned} H(Y|Z) - I(X; Y) &\geq H(Y|Z, \tilde{Z}) - I(X; Y) \\ &= I(X; Z, \tilde{Z}|Y) \\ &\geq \inf_{Z': Z' \perp\!\!\!\perp X, H(Y|X, Z')=0} I(X; Z'|Y). \end{aligned}$$

2) *Monotonicity.* Let Z satisfies $Z \perp\!\!\!\perp (X_1, X_2)$ and $H(Y_1, Y_2|X_1, X_2, Z) = 0$. Note that $(Z, X_2) \perp\!\!\!\perp X_1$ and $H(Y_1|X_1, Z, X_2) = 0$. Hence

$$\begin{aligned} I(X_1, X_2; Z|Y_1, Y_2) &\geq I(X_1; Z|X_2, Y_1, Y_2) \\ &= I(X_1; Z|X_2, Y_1, Y_2) + I(X_1; Y_2|X_2, Y_1) \\ &= I(X_1; Z|X_2, Y_1) + I(X_1; Y_2|X_2, Y_1, Z) \\ &\geq I(X_1; Z|X_2, Y_1) \\ &= I(X_1; Z|X_2, Y_1) + I(X_1; X_2|Y_1) \\ &= I(X_1; Z, X_2|Y_1) \\ &\geq \Psi(X_1 \rightarrow Y_1). \end{aligned}$$

3) *Subadditivity.* let Z_1, Z_2 satisfies $Z_i \perp\!\!\!\perp X_i$ and $H(Y_i|X_i, Z_i) = 0$, then

$$\begin{aligned} \Psi((X_1, X_2) \rightarrow (Y_1, Y_2)) &\leq I(X_1, X_2; Z_1, Z_2 | Y_1, Y_2) \\ &= I(X_1; Z_1|Y_1) + I(X_2; Z_2|Y_2). \end{aligned}$$

4) *Data processing of $\Psi + I$.* let $Z \perp\!\!\!\perp X_1$, and let $Y_2 = g(Y_1, W)$ be the functional representation of Y_2 . Then $(Z, W) \perp\!\!\!\perp X_2$, and by the the alternative characterization,

$$\begin{aligned} \Psi(X_2 \rightarrow Y_2) + I(X_2; Y_2) &\leq H(Y_2|Z, W) \\ &= H(Y_2|Z, W, Y_1) + I(Y_1; Y_2|Z, W) \\ &\leq H(Y_1|Z, W) \\ &= H(Y_1|Z). \end{aligned}$$

5) The upper bound by common entropy is a direct consequence of the data processing inequality in the previous part.

- 6) *Conditioning.* Assume that $H(Q|X) = 0$, $Z \perp\!\!\!\perp X$ and $H(Y|X, Z) = 0$, then $Z \perp\!\!\!\perp X|\{Q = q\}$ and $H(Y|X, Z, Q = q) = 0$ for all q , hence

$$\begin{aligned} I(X; Z|Y) &\geq I(X; Z|Y, Q) \\ &= \mathbf{E}_{q \sim P_Q} [I(X; Z|Y, Q = q)] \\ &\geq \mathbf{E}_{q \sim P_Q} [\Psi(X \rightarrow Y | Q = q)]. \end{aligned}$$

To show the equality case, assume $H(Q|Y) = 0$. Let \tilde{Z} satisfies $\tilde{Z} \perp\!\!\!\perp X|\{Q = q\}$ and $H(Y|X, \tilde{Z}, Q = q) = 0$ for all q . By functional representation lemma, let $\tilde{Z} = g(Q, Z)$, $Z \perp\!\!\!\perp Q$, and since we are invoking functional representation lemma over the marginal distribution of (Q, \tilde{Z}) , we can assume $Z \perp\!\!\!\perp (X, Y)|(Q, \tilde{Z})$. Hence $Z \perp\!\!\!\perp X$. We have

$$\begin{aligned} \mathbf{E}_{q \sim P_Q} [I(X; \tilde{Z}|Y, Q = q)] &= I(X; \tilde{Z}|Y, Q) \\ &= I(X; Z|Y, Q) \\ &= I(X; Z|Y) \\ &\geq \Psi(X \rightarrow Y). \end{aligned}$$

- 7) *Successive minimization.* Assume that $V \perp\!\!\!\perp X$, and let \tilde{Z} satisfy $\tilde{Z} \perp\!\!\!\perp X|\{V = v\}$ and $H(Y|X, \tilde{Z}, V = v) = 0$ for all v , then $X \perp\!\!\!\perp (\tilde{Z}, V)$. We have

$$\begin{aligned} \mathbf{E}_{q \sim P_Q} [I(X; \tilde{Z}|Y, V = v)] &= I(X; \tilde{Z}|Y, V) \\ &= I(X; \tilde{Z}, V|Y) - I(X; V|Y) \\ &= I(X; \tilde{Z}, V|Y) - I(X; V|Y) \\ &\geq \Psi(X \rightarrow Y) - I(X; V|Y). \end{aligned}$$

Note that $I(X; V|Y) + \Psi(X \rightarrow Y | V) = \Psi(X \rightarrow Y)$ if $V = \emptyset$. Also note that

$$\begin{aligned} \inf_{V: V \perp\!\!\!\perp X} \{I(X; V|Y) + \Psi(X \rightarrow Y | V)\} &\leq \inf_{V: V \perp\!\!\!\perp X, H(Y|X, Z)=0} \{I(X; V|Y) + \Psi(X \rightarrow Y | V)\} \\ &= \inf_{V: V \perp\!\!\!\perp X, H(Y|X, Z)=0} I(X; V|Y) \\ &= \Psi(X \rightarrow Y). \end{aligned}$$

■

Remark 2. If $\Psi(X, Y) = 0$, then it means that there exists Z such that $Z \perp\!\!\!\perp X$, $Z \perp\!\!\!\perp X|Y$, $H(Y|Z) = I(X; Y)$ and $H(Y|X, Z) = 0$. This implies there exists z such that $H(Y|Z = z) \geq I(X; Y)$ and $H(Y|X, Z = z) = 0$. Hence it is possible to perform one-shot zero error channel coding on the channel $P_{X|Y}$ with input distribution $P_{Y|Z=z}$ to communicate a message with entropy $\geq I(X; Y)$.

APPENDIX

PROOF OF THE BOUND ON ENTROPY IN THEOREM 1

Proposition 4. Let $\Theta \in \{1, 2, \dots\}$ be a random variable, then

$$H(\Theta) \leq \mathbf{E} [\log \Theta] + \log (\mathbf{E} [\log \Theta] + 1) + 1.$$

Proof: Let $q(\theta) = c\theta^{-\lambda}$ where $\lambda = 1 + 1/\mathbf{E} [\log \Theta]$, and $c > 0$ such that $\sum_{\theta=1}^{\infty} q(\theta) = 1$. Note that

$$\sum_{\theta=1}^{\infty} \theta^{-\lambda} \leq 1 + \int_1^{\infty} \theta^{-\lambda} d\theta = 1 + \frac{1}{\lambda - 1}.$$

Therefore

$$\begin{aligned} H(\Theta) &\leq \sum_{\theta=1}^{\infty} p_{\Theta}(\theta) \log \frac{1}{q(\theta)} \\ &= \sum_{\theta=1}^{\infty} p_{\Theta}(\theta) (\lambda \log \theta - \log c) \\ &= \lambda \mathbf{E} [\log \Theta] + \log \left(\sum_{\theta=1}^{\infty} \theta^{-\lambda} \right) \\ &\leq \lambda \mathbf{E} [\log \Theta] + \log \left(1 + \frac{1}{\lambda - 1} \right) \\ &= \mathbf{E} [\log \Theta] + \log (\mathbf{E} [\log \Theta] + 1) + 1. \end{aligned}$$

REFERENCES

- [1] A. El Gamal and Y.-H. Kim, *Network information theory*. Cambridge university press, 2011.
- [2] B. Hajek and M. Pursley, "Evaluation of an achievable rate region for the broadcast channel," *IEEE Transactions on Information Theory*, vol. 25, no. 1, pp. 36–46, Jan 1979.
- [3] F. Willems and E. van der Meulen, "The discrete memoryless multiple-access channel with cribbing encoders," *IEEE Transactions on Information Theory*, vol. 31, no. 3, pp. 313–327, May 1985.
- [4] P. Harsha, R. Jain, D. McAllester, and J. Radhakrishnan, "The communication complexity of correlation," *IEEE Trans. Info. Theory*, vol. 56, no. 1, pp. 438–449, Jan 2010.
- [5] C. E. Shannon, "Coding theorems for a discrete source with a fidelity criterion," in *IRE Int. Conv. Rec.*, 1959, vol. 7, part 4, pp. 142–163, reprint with changes (1960). In R. E. Machol (ed.) *Information and Decision Processes*, pp. 93–126. McGraw-Hill, New York.
- [6] A. El Gamal and T. M. Cover, "Achievable rates for multiple descriptions," *IEEE Trans. Inf. Theory*, vol. 28, no. 6, pp. 851–857, 1982.
- [7] Z. Zhang and T. Berger, "New results in binary multiple descriptions," *IEEE Trans. Inf. Theory*, vol. 33, no. 4, pp. 502–521, 1987.
- [8] R. M. Gray and A. D. Wyner, "Source coding for a simple network," *Bell Syst. Tech. J.*, vol. 53, no. 9, pp. 1681–1721, 1974.
- [9] S. I. Gelfand and M. S. Pinsker, "Coding for channel with random parameters," *Probl. Control Inf. Theory*, vol. 9, no. 1, pp. 19–31, 1980.
- [10] J. Liu, P. Cuff, and S. Verdú, "Resolvability in E_γ with applications to lossy compression and wiretap channels," in *2015 IEEE International Symposium on Information Theory (ISIT)*, June 2015, pp. 755–759.
- [11] N. Datta, J. M. Renes, R. Renner, and M. M. Wilde, "One-shot lossy quantum data compression," *IEEE Transactions on Information Theory*, vol. 59, no. 12, pp. 8057–8076, Dec 2013.
- [12] S. Verdú, "Non-asymptotic achievability bounds in multiuser information theory," in *Communication, Control, and Computing (Allerton), 2012 50th Annual Allerton Conference on*, Oct 2012, pp. 1–8.
- [13] J. Liu, P. Cuff, and S. Verdú, "One-shot mutual covering lemma and marton's inner bound with a common message," in *2015 IEEE International Symposium on Information Theory (ISIT)*, June 2015, pp. 1457–1461.
- [14] S. Watanabe, S. Kuzuoka, and V. Y. F. Tan, "Non-asymptotic and second-order achievability bounds for source coding with side-information," in *2013 IEEE International Symposium on Information Theory*, July 2013, pp. 3055–3059.
- [15] M. H. Yassaee, M. R. Aref, and A. Gohari, "A technique for deriving one-shot achievability results in network information theory," in *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, July 2013, pp. 1287–1291.
- [16] J. T. Pinkston, *Encoding independent sample information sources*. Research Laboratory of Electronics, Massachusetts Inst. of Technology, 1967.
- [17] M. Pursley and L. Davison, "Variable rate coding for nonergodic sources and classes of ergodic sources subject to a fidelity constraint," *IEEE Transactions on Information Theory*, vol. 22, no. 3, pp. 324–337, May 1976.
- [18] K. Mackenthun and M. Pursley, "Variable-rate universal block source coding subject to a fidelity constraint," *IEEE Transactions on Information Theory*, vol. 24, no. 3, pp. 349–360, May 1978.
- [19] O. Kosut and L. Sankar, "Universal fixed-to-variable source coding in the finite blocklength regime," in *2013 IEEE International Symposium on Information Theory*, July 2013, pp. 649–653.
- [20] R. Ahlswede and J. Körner, "Source coding with side information and a converse for degraded broadcast channels," *IEEE Trans. Inf. Theory*, vol. 21, no. 6, pp. 629–637, 1975.
- [21] A. D. Wyner and J. Ziv, "The rate–distortion function for source coding with side information at the decoder," *IEEE Trans. Inf. Theory*, vol. 22, no. 1, pp. 1–10, 1976.
- [22] C. H. Bennett, P. W. Shor, J. Smolin, and A. V. Thapliyal, "Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem," *IEEE Trans. Info. Theory*, vol. 48, no. 10, pp. 2637–2655, 2002.
- [23] P. Cuff, "Distributed channel synthesis," *IEEE Trans. Info. Theory*, vol. 59, no. 11, pp. 7071–7096, 2013.
- [24] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor, and A. Winter, "The quantum reverse shannon theorem and resource tradeoffs for simulating quantum channels," *IEEE Trans. Info. Theory*, vol. 60, no. 5, pp. 2926–2959, May 2014.
- [25] Z. Zhang, E. h. Yang, and V. K. Wei, "The redundancy of source coding with a fidelity criterion. 1. known statistics," *IEEE Transactions on Information Theory*, vol. 43, no. 1, pp. 71–91, Jan 1997.
- [26] D. S. Ornstein and P. C. Shields, "Universal almost sure data compression," *The Annals of Probability*, pp. 441–452, 1990.
- [27] R. Venkataramani, G. Kramer, and V. K. Goyal, "Multiple description coding with many channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 9, pp. 2106–2114, Sep. 2003.
- [28] J. Wang, J. Chen, L. Zhao, P. Cuff, and H. H. Permuter, "On the role of the refinement layer in multiple description coding and scalable coding," *IEEE Trans. Inf. Theory*, vol. 57, no. 3, pp. 1443–1456, Mar. 2011.
- [29] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Inf. Theory*, vol. 25, no. 3, pp. 306–311, 1979.
- [30] J. F. C. Kingman, *Poisson processes*. Oxford University Press, 1992.
- [31] G. R. Kumar, C. T. Li, and A. El Gamal, "Exact common information," in *Proc. IEEE Symp. Info. Theory*. IEEE, 2014, pp. 161–165.
- [32] C. T. Li and A. El Gamal, "Distributed simulation of continuous random variables," in *Proc. IEEE Symp. Info. Theory*, July 2016, pp. 565–569.